Ministry of Education and Science of Ukraine
Donbass State Engineering Academy

# FUNCTIONAL ANALYSIS

## Methodical instructions
## to classes and independent work

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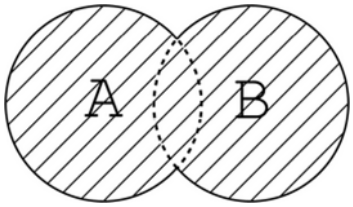
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# 1 THE CONCEPT OF A SET. KINDS OF SETS. POTENCY

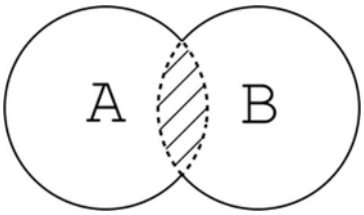
## Basic theoretical statements

Union of sets: $C = A \bigcup B$ (pic. 1).



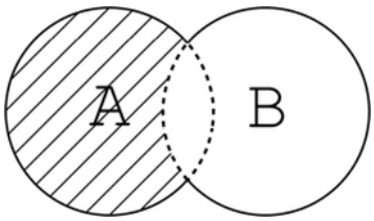

*Picture 1 – Union of sets*

Intersection: $C = A \bigcap B$ (pic.2).



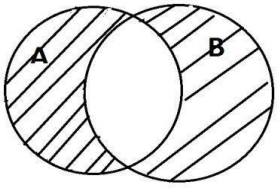

*Picture 2 – Intersection*

Difference: $C = A / B$ (pic. 3).



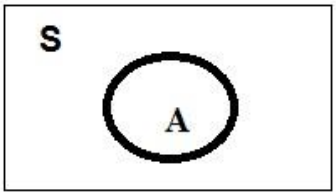

*Picture 3* – Difference

Symmetrical difference: $C = A \Delta B$ (pic. 4).




*Picture 4 – Symmetrical difference*

Complement $S / A$ (pic. 5).




*Picture 5 – Complement*

***Ex. 1.1***. Perform operations on sets.
$A = \{2,5,7,9\}$, $B = \{3,5,8,9,12\}$.

Solution:

$$A \cup B = \{2,3,5,7,8,9,12\},$$
$$A \cap B = \{5,9\},$$
$$A / B = \{2,7,\}.$$

***Ex. 1.2***. Perform operations on sets
$X = \{x : -3 \le x \le 8\}$, $Y = \{x : -5 \le x \le 6\}$.

Solution:

$$X \cap Y = \{x : -3 \le x < 6\},$$
$$X \cup Y = \{x : -5 < x \le 8\}.$$

***Ex. 1.3***. Find the intersection of such sets:
a) $X$ – set of two-digit natural numbers, $Y$ – set of paired natural numbers;
b) $X$ – set of all the parallelograms, $Y$ – set of trapezoids.

Solution:
a) $X = \{10,11,...,99\}$, $Y = \{2,4,...,98,...\}$, $X \cap Y = \{10,12,14,...,98\}$;
b) $\emptyset$ $\emptyset$.

***Ex. 1.4***. Set a set:

$$A = \{10\},\ B = \{10,15\},\ C = \{5,10,15\}.$$

Define the entry of one set to another.

Solution:

$$A \in B,\ C \in A,\ C \in B.$$

***Ex. 1.5***. Prove the properties of operations on sets:

a) $A \bigcup B = B \bigcup A$;

b) $A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C)$.

Solution:

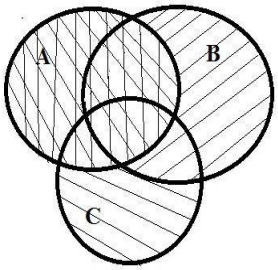
a) Let $x \in A \bigcup B$, then $x \in A$ or $x \in B$. Here from $x \in B \bigcup A$. The reverse is proved in a similar way.

b) Let $\epsilon A \cup (B \cap C)$, $x \in A \bigcup (B \bigcap C)$ then or $x \in A$, or $x \in (B \bigcap C)$, $x \in (A \bigcup B)$ і $(A \bigcup C)$. Then $x \in A \bigcup B$, $x \in A \bigcap C$. So, $x \in B \bigcup C$, $x \in A$, $x \in B \bigcup C$. $x \in A \bigcup B$ і $x \in A \bigcap B$.

$(A \bigcup B) \bigcap (A \bigcup C)$, $x \in (A \bigcup B) \bigcap (A \bigcup C)$.

Prove the reversal.

Let $x \epsilon (A \cup B) \cap (A \cup C)$, то $x \epsilon (A \cup B)$ и $x \epsilon (A \cup C)$ $x \in (A \bigcup B) \bigcap (A \bigcup C)$, then $x \in (A \bigcup B)$ і $x \in (A \bigcup C)$. So, $x \in A \bigcup B$ і $x \in A \bigcup C$. It follows from this, that $x \epsilon A$ или $x \epsilon B \cap C$, $x \epsilon A \cup (B \cap C)$ $x \in A$, or $x \in B \bigcap C$, $x \in A \bigcup (B \bigcap C)$ (pic. 6).




*Picture 6 – Proof of Statement*

***Ex. 6***. Find the complement to the set of the natural even numbers

$$A = \{x : x + 2k,\ k \in N\}.$$

Solution:

$$N / A = \{x : x = 2k + 1,\ k \in N\}.$$

***Ex. 1.7***. Prove that the set is countable:

$$\epsilon N : n = k^2, k \epsilon N\} \{n \in N : n = k^2,\ k \in N\}.$$

Solution: To the numbers 1, 4, 9, 16, 25, 36... we substitute the next numbers 1, 2, 3, 4, 5, 6,... So, the set is countable.

***Ex. 1.8***. Prove that the set of positive rational numbers is countable.

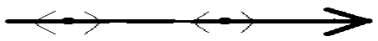
$$R=\left\{1,2,3,4,5,6,7,...,\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2},...,\frac{1}{3},\frac{2}{3}...\right\}$$

Solution: The next sets are countable:

$$0,1,2,3,4,5,6,7,...$$
$$\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2},...,$$
$$\frac{1}{3},\frac{2}{3},\frac{4}{3},\frac{5}{3},\frac{7}{3},...,$$
$$\frac{1}{4},\frac{3}{4},\frac{5}{4},...\cdot$$

The union of countable sets is countable.

***Ex. 1.9***. Prove that the set of segments on a line that do not intersect in pairs is finite or countable (pic. 7).



*Picture 7 – Numerical sets*

Solution: Each segment has a rational point and for segments that do not intersect these points, are different. So, the set is countable.

## 2 METRIC SPACES

### *Basic theoretical statements*

Metric is a positive function that satisfies two conditions:

1. $\rho(x,y)=0$ if and only if $x=y$ (axiom of identity).
2. $\rho(x,y)=\rho(y,x)$ (axiom of symmetry).
3. $\rho(x,z)\le\rho(x,y)+\rho(y,z)$ (axiom of triangle).

The mapping of sets:
$f:M\to N,\ a\in M,\ b\in N,\ b=f(a),\ a$ – prototype, $b$ – type.

If $f(M)=N$ Then the mapping is a surjection.

If the condition $x \neq x_2$ follows $y_1 = f(x_1) \neq y_2 = f(x_2)$ Image mapping is an injection.

Together the subjection and injection are bijection.

*Classification of metric space points*. Point $x_0$ is called the boundary point of the set of $M$ & $X$ ($M$ is a subset of space $X$), if any of its environments contains at least one point of the set $M$.

*The inner point* of a set is the point that enters this set with its surroundings.

*An isolated point* of a set is a point at which there is an environment that does not contain other points of this set.

***Ex. 2.1.*** Prove that the given functions on the specified sets are metrics:

a) $\rho(x,y)=\min\{1,|x-y|\}$ on set $R$;

b) $\rho(x,y)=\sum_{n=1}^{\infty}\frac{|x_n-y_n|}{n}$ on set $l_2$.

Solution:

a) Let's check the fulfillment of the axioms of the metric:

1. $x=y,\ \rho(x,y)=\min\{1,|x-x|\}=0\cdot$
2. $\rho(x,y)=\min\{1,|x-y|\}=\min\{1,|y-x|\}=\rho(y,x),$
3. $\rho(x,z)=\min\{1,|x-z|\}=\min\{1,|x-y+y-z|\}\le\rho(x,y)+\rho(y,z).$

All axioms are fulfilled.

b) Check the fulfillment of the axioms of the metric

1. $x=y,\ \rho(x,y)=\sum_{n=1}^{\infty}\frac{|x_n-x_n|}{n}=0\cdot$
2. $\rho(x,y)=\sum_{n=1}^{\infty}\frac{|x_n-y_n|}{n}=\sum_{n=1}^{\infty}\frac{|y_n-x_n|}{n}=\rho(y,x)\cdot$
3. $\rho(x,z)=\sum_{n=1}^{\infty}\frac{|x_n-z_n|}{n}=\sum_{n=1}^{\infty}\frac{|x_n-y_n+y_n-z_n|}{n}\le\rho(x,y)+\rho(y,z)\cdot$

All axioms are fulfilled.

***Ex. 2.2.*** Prove that the function

$$\rho(x,y)=\max_{1\le n\le m}|x_n-y_n|$$

on set $R^n$ is a metric.

Solution: Let's check the fulfillment of the axioms of the metric.

1. $x = y$, $\rho(x,y) = \max\limits_{1\le n\le m}|x_n - x_n| = 0$.

2. $\rho(x,y) = \max\limits_{1\le n\le m}|x_n - y_n| = \max\limits_{1\le n\le m}|y_n - x_n| = \rho(y,x)$.

3. $\rho(x,z) = \max\limits_{1\le n\le m}|x_n - z_n| \le \max\limits_{1\le n\le m}|x_n - y_n| + \max\limits_{1\le n\le m}|y_n - z_n| \le \rho(x,y) + \rho(y,z)$.

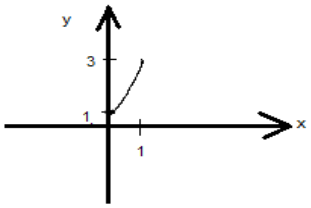
The axioms of the metric are fulfilled.

***Ex. 2.3.*** Which of the following display functions is surjection, injection, bijection?

a) $f(x) = 3^x$, b) $f(x) = 12(x - 1/2)^2$, c) $f(x) = 3x$.

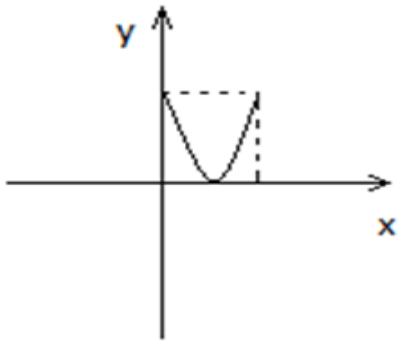
Solution:

a) $y \in [0,1]$ – there is no solution, the reflection is not surjection, injection (pic.8).



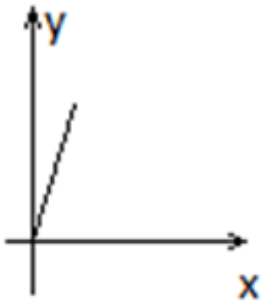

*Picture 8 – The graph of function*

b) $y_1 = y_2$, for example, when $x_1 = 1$, $x_1 = -1$, it's not a subjection, it's not an injection (pic. 9).




*Picture 9 – The graph of function*

c) The given function is injection and surjection, therefore – bijection (pic. 10).




*Picture 10 – The graph of function*

***Ex. 2.4.*** Prove that when

$$f : E \to F,\ A \subset F,\ B \subset F,$$

then

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B).$$

Solution: If $x \in f^{-1}(A \cap B)$, then $f(x) \in A$ і $f(x) \in B$. Here from $x \in f^{-1}(A)$ і $x \in f^{-1}(B)$. So, $x \in f^{-1}(A) \cap f^{-1}(B)$.

***Ex. 2.5.*** Specify boundary, internal, and isolated points of sets:
a) $A = [0;3) \cup \{6\}$, b) $E = \{1;1/4;1/16;...\}$.

Solution:
a) boundary $[0;3)$, internal $[0;3)$, isolated $\{6\}$.
b) boundary $\{0\}$, no internals, isolated $\{1;1/4;1/16;...\}$.

***Exercises for independent work***

***Ex. 2.6.*** Check whether the functions are metrics on the specified sets:
а) $\rho(x,y) = |x^3 - y^3|,\ R$;
б) $\rho(x,y) = |x^2 - y^2|,\ X = [0;\infty)$.

***Ex. 2.7.*** Classify the points of sets:
а) $A = [0;3) \cup (4;5) \cup \{-1\} \cup \{10\}$; б) $A = \{(x_1, x_2),\ x_{1,2} \in Q\}$.

***Ex. 2.8.*** Specify a set that has exactly three boundary points.

# 3 OPEN AND CLOSED SETS

***Basic theoretical statements***

Point $x_0$, which belongs to the space $X$, is called the limit of a sequence $\{x_n\} = x_1, x_2, ..., x_n ...$ If for any arbitrarily small number $\varepsilon > 0$ exists the number $N$ such that, starting with it, all sequence numbers are included in the $\varepsilon$ -round the point $x_0$.

Point $x_0$ is called the limit of a sequence $\{x_n\} = x_1, x_2, ..., x_n ...$, in case if $\lim\limits_{n\to\infty} \rho(x_0, x_n) = 0$.

A set is said to be *closed* if it coincides with its closure (ie all boundary points are contained in this set).

An *open* set is one that consists only of internal points.

***Ex. 3.1.*** Find the limit.

$$\{x_n\} = \left\{\frac{1}{n^2}\right\} = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, ..., \right\}.$$

Solution: Prove by definition that the number 0 is the limit of this sequence.

I way:

$$\rho(0, x_N) < \varepsilon,$$

$$\rho(0, x_N) = |0 - x_N| < \varepsilon => |x_N < \varepsilon|,$$

$$\frac{1}{n^2} < \varepsilon,$$

$$n^2 > \frac{1}{\varepsilon},$$

$$n > \frac{1}{\sqrt{\varepsilon}}, \; N = \left[\frac{1}{\sqrt{\varepsilon}}\right].$$

For example, if $\varepsilon = 100$, then $N = 10$.

II way:

$$\lim_{n\to\infty} \left|0 - \frac{1}{n^2}\right| = \lim_{n\to\infty} \frac{1}{n^2} = \frac{1}{\infty} = 0.$$

***Ex. 3.2.*** Specify closed and open sets:

1. $A = [0;3) \cup \{6\}$.

2. $A=[0;3)$.
3. $A=(-2;1)$.
4. $A=[0;7]$.

Solution:
1. Open.
2. Open.
3. Open.
4. Closed.

***Exercises for independent work***

***Ex. 3.3.*** Prove that the set $A=(-5;3)\cap[0;1]\cup\{2\}$ is a closed one.

## 4 COMPLETE AND COMPACT METRIC SPACES

***Basic theoretical statements***

Let $X$ be the metric space and $\{x_n\}$ be the sequence of elements of this space. A sequence $\{x_n\}$ is said to be *fundamental* if there is a number $N$, from which for any number $m > N$ the inequality is fulfilled $\rho(x_n, x_m) < \varepsilon$ as for any small $\varepsilon > 0$.

The space in which each fundamental sequence coincides (up to the point of this space) is called *complete*.

Let $X$ be arbitrary metric space. The set $M \in X$ is said to be compact, if a convergent sequence $\{x_n\}$ can be distinguished from any sequence $\{x_n\}$ of this set.

***Exercises for independent work***

***Ex. 4.1.*** Check that the sequence $\{f_n(x)\}=\frac{1}{x^n}$ is fundamental in space $C_{2[1;3]}$.

# 5 THE PRINCIPLE OF COMPRESSIVE MAPPINGS

***Basic theoretical statements***

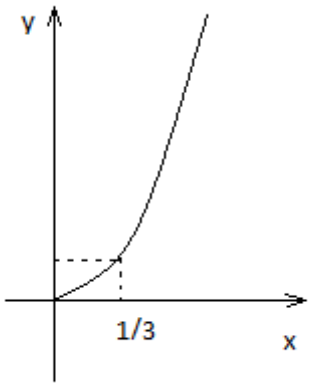
Let $X$ be the metric space and $f(x)$ – reflection of metric space. The mapping $f(x)$ is called a compression map (compression) if for any two points $x,\ y$, that belong with $X$, there is a number $\alpha < 1$ such that the inequality holds.

$$\rho(f(x), f(y)) \le \rho(x; y).$$

$f(x) = x$ – nerve point.

***Ex. 5.1.*** Show that the function $f(x) = x^2$ displays a segment $\left[0, \frac{1}{3}\right]$. Is this mapping compressed?

Solution: Make a drawing (pic. 11).




*Picture 11 – The graph of function*

Because $0 \le x^2 \le \frac{1}{9}$, then $\left[0, \frac{1}{9}\right] \subset \left[0, \frac{1}{3}\right]$ · Displays a segment. Let $c \in [x, y] \subset \left[0, \frac{1}{3}\right]$. Because

$$|f'(x)| = 2c,$$

$$0 \le ac \le \frac{2}{3}, \quad a < 1$$

$$0 \le c \le \frac{1}{3},$$

Mapping is compressed.

***Ex. 5.2.*** Is the compressive reflection $f(x)=x+\frac{1}{x}$ of the semi-straight line $[1,\infty)$ in itself?

Solution:

$$|f'(x)|=|1-1/x^2|,\cdot$$

$$1\le c^2\le\infty,$$

$$0\le\frac{1}{c^2}\le 1^{,}$$

$$1-\frac{1}{c^2}\in[0,1]^{\cdot}$$

When $c$ is large enough, the parameter $\alpha$ becomes arbitrarily close to unity, so the display is not compressible.

***Ex. 5.3.*** Show that the equation $x=\sqrt[3]{x+2}$ can be solved by the method of successive approximation and calculate its roots to the nearest $0{,}01$.

Solution: It must be proved that the equation specifies the compression mapping. By Banach fixed-point theorem:

1. The display translates the segment $[a;b]$ in itself.

2. Lipschitz continuity is fulfilled.

$$\exists f'(x) \text{ on } [a;b] \text{ i } |f'(x)|\le k<1\cdot$$

We have (pic. 12)

$$f_1(x)=xf_2(x)\sqrt[3]{x+2}\cdot$$

$\bar{x}\in[0,2]$

1. $x=f(x),f(x)\sqrt[3]{x+2}\cdot$

2. $f(x)$ displays $[0,2]$ in itself.

$$0\le x\le 2,$$

$$2\le x+2\le 4,$$

$$\sqrt[3]{2}\le\sqrt[3]{x+2}\le\sqrt[3]{4},$$

$$[\sqrt[3]{2},\sqrt[3]{4}]\subset[0,2]\cdot$$

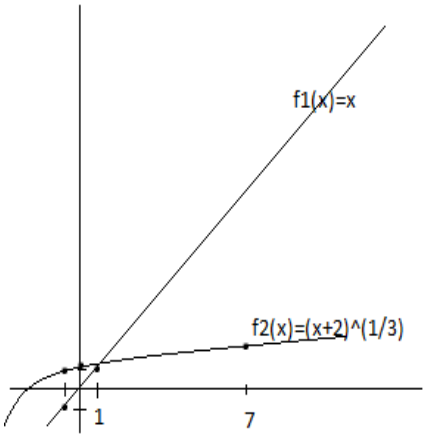
$$f'(x)=\frac{1}{3}(x+2)^{-\frac{3}{2}}=\frac{1}{3*\sqrt[3]{(x+2)^2}},$$

$$\sqrt[3]{4}\le\sqrt[3]{(x+2)^2}\le\frac{\sqrt[3]{16}}{2*\sqrt[3]{2}},$$

$$\frac{1}{6*\sqrt[3]{2}}\le f'(x)\le\frac{1}{3*\sqrt[3]{4}}\le\frac{1}{3},$$

$$|f'(x)|\le\frac{1}{3}<1.$$

This means that the mapping is compressed.




*Picture 12 – The graphs of functions*

$$x_0=1\quad f(x_0)=\sqrt[3]{1+2}=1{,}442+x_1,$$

$$x_1=1{,}442\quad f(x_1)=\sqrt[3]{1{,}442+2}=1{,}510+x_2,$$

$$x_2=1{,}510\quad f(x_2)=1{,}520+x_3,$$

$$x_3=1{,}520\quad f(x_3)=1{,}521\ .$$

So, the solution: $\bar{x}=1{,}52\,\bar{x}=1{,}52$.

***Exercise for independent work***

***Ex. 5.4.*** Show that the equation $xe^x=1$ specifies the compression mapping.

# 6 LINEAR AND NORMALIZED SPACES

***Basic theoretical statements***

Properties of the norm:

1. $\|x\| \geq 0$, moreover $\|x\| = 0$ only if $x = 0$.
2. $\|ax\| = |a| \cdot \|x\|$, $a$ is a number.
3. $\|x + y\| \leq \|x\| + \|y\|$.

***Ex. 6.1.*** Prove that $\|x\|_1 = \sum_{i=1}^{n} \|x_i\|$ determines the norm in space $R^n$.

Solution: Let's check performance of properties of norm:

1. $\|x\|_1 = \sum_{i=1}^{n} |x_i| \sum_{i=1}^{n} |x_i| \geq 0 \geq 0$,
2. $\|\alpha x\|_1 = \sum_{i=1}^{n} |\alpha x_i| = |\alpha| \sum_{i=1}^{n} |x_i| = |\alpha| \|x\|_1$,
3. $\|x + y\|_1 = \sum_{i=1}^{n} |x_i + y_i| \leq \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i| = \|x\|_1 + \|y\|_1$.

***Ex. 6.2.*** Calculate the norms $\|x\|, \|x\|_1, \|x\|_\infty$ of an element $c = (1;0;-7;2)$ in space $R^n$.

Solution:

$$\|x\| = \sqrt{52},$$

$$\|x\|_1 = \sqrt{10},$$

$$\|x\|_\infty = 7.$$

***Ex. 6.3.*** Calculate the norm of element $x(t) = 2t - t^2 + 1$ in space $C_{[-1;2]}$.

Solution:

$$\|x\| = \max_{-1 \leq t \leq 2} |x(t)| = \max_{-1 \leq t \leq 2} |2t - t^2 + 1| = 2.$$

***Ex. 6.4.*** Find the norm of the element in the specified space:

a) $\cos \pi t$ in space $C_{[0;1]}$; b) $2^t$ in space $C_{[-1;5]}$.

Solution:

a) $\|x\| = \max\limits_{a \le t \le b} |f(t)| = \max\limits_{0 \le t \le 1} |\cos \pi t| = 1$.

b) $\|2^t\| = \max\limits_{-1 \le t \le 5} |2^t| = 2^5 = 32$.

***Exercise for independent work***

***Ex. 6.5***. Prove that in space $R^n$ the norm can also be specified by a formula $\|x\|_\infty = \max |x_i|,\ 1 \le i \le n$.

## 7 EUCLIDEAN SPACES

***Basic theoretical statements***

Properties of a scalar product:

1. $(x, y) = (y, x)$.
2. $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$.
3. $(\lambda x, y) = \lambda (x, y)$.
4. $(x, x) \ge 0$, moreover $(x, x) = 0$ only if $x = 0$.

***Ex. 7.1***. Calculate the scalar product of vectors $\bar{a}(2; -4; 5; 1)$ $\bar{a}(2;–4;5;1)$ $\bar{b}(0;2;3;0)$.

Solution:

$$(\bar{a}, \bar{b}) = 2 \cdot 0 + (-4) \cdot 2 + 5 \cdot 3 + 1 \cdot 0 = 7,\ \bar{a},\ \bar{b} \in R^4.$$

***Ex. 7.2.*** Calculate the scalar product of elements $f(t) = t$ i $g(t) = \frac{1}{t^2}$ in space $C_{2[1,3]}$.

Solution:

$$(f, g) = \int_1^3 t \cdot \frac{1}{t^2} dt = \int_1^3 \frac{1}{t} dt = \ln|t| \Big|_1^3 = \ln|3| - \ln|1| = \ln 3.$$

**Ex. 7.3** Calculate the scalar product of elements $x(t)=t$, $y(t)=\frac{1}{t}$ in space $C_{2[1;2]}$.

Solution:

$$(x,y)=\int_1^2 t\cdot\frac{1}{t}dt=1.$$

**Ex. 7.4** Calculate the scalar product of elements $s_1=\{1;\frac{1}{3};\frac{1}{9}...\}$, $s_2=\{0;\frac{3}{7};\frac{9}{49};...\}$ in space $l_2$.

Solution:

$$(s_1,s_2)=\frac{1}{7}+\frac{1}{49}+...=\frac{1}{6}.$$

***Ex. 7.5***. Find the scalar product of elements $x=(1;\frac{1}{2};\frac{1}{4};\frac{1}{8};...)$, $y=(1;\frac{1}{3};\frac{1}{9};\frac{1}{27};...)$ in space $l_2$.

Solution:

$$(x,y)=\left(1*1+\tfrac{1}{2}*\tfrac{1}{3}+\tfrac{1}{4}*\tfrac{1}{9}+\cdots\right)(x,y)=1\cdot1+\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{9}+...;$$

$$(x,y)=\sum_{n=0}^{\infty}\frac{1}{2^n}\cdot\frac{1}{3^n}=\sum_{n=0}^{\infty}\frac{1}{6^n},$$

$$\sum_{n=0}^{\infty}\frac{1}{6^n}=\frac{1}{1-\frac{1}{6}}=\frac{6}{5}.$$

***Exercises for independent work***

***Ex. 7.6.*** Calculate the scalar product of elements $x(t)=t$, $y(t)=\sin t$ in space $C_{2[0;1]}$.

***Ex. 7.7.*** Calculate the scalar product of elements $a=(-1;0;5)$, $b=(2;4;3)$ in space $R^3$.

# 8 HILBERT SPACES

***Basic theoretical statements***

*Fourier series in Hilbert space.*

Kind of expression $\sum C_k \varphi_k \sum_{k=1}^{\infty} c_k \varphi_k$ , where $\varphi_1, \varphi_2 ..., \varphi_n, ...$ – orthogonal normalized system and $c_k = (f, \varphi_k),\ k = 1,2,..., n$ – Fourier coefficients, and this kind of expression is called Fourier series.

***Ex. 8.1.*** For a given element, construct a Fourier series according to the orthogonal systems of the specified space:

a) $a = (1;2/3;4/9;...), a \in l_2$; b) $f(x) = x + 1,\ C_{2[-\pi;\pi]}$.

Solution:

a) Let's calculate the Fourier coefficients of a given element according to the formula

$$c_k = (f, \varphi_k),\ k = 1,2,... .$$

We receive

$$c_1 = 1, .\ c_2 = 2/3,\ c_3 = 4/9, ... .$$

Let's create a Fourier series of elements $a = (1;2/3;4/9;...),\ a \in l_2$:

$$c_1\varphi_1 + c_2\varphi_2 + ... = \sum_{k=1}^{\infty} c_k\varphi_k = 1(1;0;0...) + 2/3(0;1;0;...) + 4/9(0;0;1;...) + ... =$$
$$= (1;2/3;4/9;...) .$$

b) Calculate the Fourier coefficients of a given element by the formula

$$c_k = (f, \varphi_k),\ k = 1,2,...$$

And receive

$$c_1 = -\frac{\pi}{24},\ c_{2n} = \frac{1-(-1)^n}{6\pi n^2},\ c_{2n} = \frac{1-(-1)^n}{6\pi n^2},$$

Create a Fourier series of elements $f(x) = x + 1$:

$$c_1\varphi_1 + c_2\varphi_2 + ... = \sum_{k=1}^{\infty} c_k\varphi_k = -\frac{\pi}{24} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{6\pi n^2} \cos nx + \frac{5(-1)^{n+1}}{6\pi} \sin nx .$$

***Ex. 8.2.*** Develop an element $I=(1/2;1/4;1/8;...)\in l_2\in e_2\in e_2$ according to the orthonormal system of this space $e_1=(1,0,0,...)$, $e_2=(0,1,0,...)$, $\cdots$.

Solution: Let's calculate the Fourier coefficients of a given element by formulas

$c_k=(f,\varphi_k),\ k=1,2,...$

And receive

$$c_1=1/2,\ c_2=1/4,\ c_3=1/8,...$$

Create the Fourier series $I=(1/2;1/4;1/8;...)\in l_2$:

$$c_1\varphi_1+c_2\varphi_2+...=\sum_{k=1}^{\infty}c_k\varphi_k=$$

$$=\sum_{k=1}^{\infty}(1/4)^k\cdot c_k=1/2\,(1,0,0,...)+1/4\,(0,1,0,...)+1/8\,(0,0,1,...)+...$$

***Exercises for independent work***

***Ex. 8.3.*** Create a series of Fourier for a function $f(x)=\dfrac{\pi-x}{2}$ according to the orthogonal system of space $C_{2[-\pi;\pi]}$.

## 9 LINEAR FUNCTIONALS IN LINEAR NORMALIZED SPACES

***Basic theoretical statements***

Limited linear functionality

$$|f(x)|\le M\|x\|,$$

where $M$ is the smallest of these constants and the norm $\|f\|$,

$$|f(x)|\le\|f\|\|x\|.$$

***Ex. 9.1.*** Calculate the norm of functional $f(x)=x(1)-2x(2)$, given in space $C_{[0,2]}$.

Solution:

$|f(x)| = |x(1) - 2x(2)| \le |x(1) + |2x(2)|| = |x(1)| + 2|x(2)| \le 3\max|x| = 3\|x\|$

$$|f(x)| = |x(1) - 2x(2)| \le |x(1) + |2x(2)|| =$$

$$= |x(1)| + 2|x(2)| \le 3\max_{0\le x\le 2}|x| = 3\|x\| \cdot$$

Suppose that $\|f\| = 3$.

$$\|f\| \le \frac{|f(x)|}{\|x\|_{C_{[0,2]}}} = \frac{|x(1) - 2x(2)|}{\max\limits_{0\le t\le 2}|x(t)|},$$

$$\max_{0\le t\le 2}|x(t)| = 1 \cdot$$

Choose a continuous function $x(t)$, so that $\max\limits_{0\le x\le 2}|x(t)| = 1$. So, $\|f\| = 3$ is a norm of functional.

***Exercises for independent work***

**Ex. 9.2** Calculate the norm of functional $\|f\| = \int_0^1 x(t)dt$ in space $C_{2[0,1]}$.

## 10 LINEAR OPERATORS

***Basic theoretical statements***

Let $E_1$ and $E_2$ be the two linear normalized spaces. *Linear operators,* acting with $E_1$ in $E_2$, is called a mapping $A = Ax$, $x \in E_1$, that satisfies the conditions:

1. $A(x + y) = Ax + Ay$ (additivity).
2. $A(\alpha x) = \alpha Ax$ (homogeneity).

The limitation of the linear operator means that such a constant exists $M$, that

$$\|Ax\| \le M \cdot \|x\|, \ \forall x \in E_1$$

The smallest of these constants is called the norm of the operator $A$ and signed as $\|A\|$.

***Exercises for independent work***

***Ex. 10.1.*** Estimate the norm of the Fredholm integral operator in space $C_{[a;b]}$.

## 11 THE CONCEPT OF MEASURE

***Basic theoretical statements***

*Closed and open sets on a line.*

A set is called *closed*, if it coincides with its closure (all boundary points are contained in this set).

*An open* set is a set consisting only of internal points.

*The measure of the interval* $(a;b)$ is its length:

$$m(a;b) = b - a\,.$$

The measure of the segment $[a;b]$ equals its length.

***Ex. 11.1.*** Given set $A = [0;1) \cup 2$. Find its:

1. Limit points.
2. Boundary points.
3. Points of contact.
4. Internal points.
5. Isolated points.

Solution:

1. Limit points $[0;1]$.
2. Boundary points $0,1,2$.
3. Points of contact $[0;1],1$.
4. Internal points $(0;1)$.
5. Isolated point $2$.

***Ex. 11.2.*** Find the measure of these sets on the line:

1. $[0;2]$.
2. $[0;1) \cup 2$.
3. $[0;2)$.
4. $(0,1) \cup (4;5) \cup \{-1\} \cup \{10\}$.
5. $[2;5] \bigcup \{3\} \bigcup [7;8]$.
6. $\bigcup\limits_{k=1}^{\infty} E_k,\ E_k = [\frac{1}{k+1}; \frac{1}{k}]$.

Solution:

1. $m[0;2]=2$.
2. $m([0;1]\cup 2)=2$.
3. $m(0;2]=2$.
4. $m((0;3)\cup(4;5)\cup\{-1\}\cup\{10\})=3$.
5. $m([2;5]\cup(3)\cup[7;8])=4$.
6. $m\left(\bigcup_{k=1}^{\infty}E_k, E_k=[\frac{1}{k+1};\frac{1}{k}]\right)=\sum_{k=1}^{\infty}\frac{1}{k(k+1)}=1$.

***Exercises for independent work***

***Ex. 1.3.*** Find the boundary points of such sets on a numerical line. Specify closed sets.

1. $R$.
2. $\emptyset\emptyset$.
3. $Z$.
4. $[0;1]$.
5. $(0;1)$.
6. $\left\{\frac{1}{n}, n\in N\right\}$.
7. $R^+$ – positives.
8. $R^-$ – negatives.

# 12 MEASURABLE FUNCTIONS

***Basic theoretical statements***

Measurable functions on a line: $f(x)$ is measurable on the set $E$ function, if:

1. $E$ – measurable.
2. $E(f > a)$– measurable for all valid $a$.

*Measurement criterion.* In order for a finite function $f(x)$ to be measurable on a segment $[a;b]$ almost everywhere, it is necessary and sufficient for it to be continuous almost everywhere.

***Ex. 12.1.*** Specify measurable functions on a given set:

1. $f(x)=x+1,\ E=[0;3]\cdot$
2. $f(x)=x+1,\ E=[0;\infty)\cdot$
3. $f(x)=x+1,\ E=R\cdot$
4. $f(x)=\frac{1}{x},\ E=[0;3]\cdot$
5. $f(x)=\frac{1}{x},\ E=[1;3].$
6. $f(x)=5,\ E=[0;3]\cdot$

Solution:

1. Measurable.
2. Immeasurable, because the set E is immeasurable.
3. Immeasurable, because the set E is immeasurable.
4. Measurable.
5. Measurable.
6. Measurable.

***Ex. 12.2.*** Prove that the function $y=f(x)$, is measurable according to Lebesgue on $R$, if

$$f(x)=\sum_{n=1}^{\infty} arctg\frac{x}{x^4+n^4}\cdot$$

Solution: For the measurability of a function $f(x)$, it is necessary and sufficient that it can be represented as a boundary of a uniformly convergent sequence of simple, measurable functions.

The members of this series are continuous function $R$, so they are measurable according to Lebesgue. Because

$$\frac{x}{x^4+n^4}\ \frac{x}{x^4+n^4} \sim\sim \frac{x}{n^4},$$

then the given series is equivalent to the series

$$\sum_{n=1}^{\infty}\frac{x}{n^4},$$

which coincides. So, a number

$$\sum_{n=1}^{\infty} arctg\ \frac{x}{x^4+n^4}\sum_{n=1}^{\infty} arctg\ \frac{x}{x^4+n^4}$$

also coincides, so the sum of the series is measurable according to Lebesgue.

***Ex. 12.3.*** Prove that the function $y=f(x),\ x\in R$ measurable according to Lebesgue $R$, if

$$f(x)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n+\cos x}$$

Solution: The members of the series under consideration are continuous functions on $R$. Consider a series in which

$$u_n=\frac{1}{n+\cos x},$$

$$u_2=\frac{1}{2+\cos x}>u_3=\frac{1}{3+\cos x}>u_4=\frac{1}{4+\cos x}>\ldots>,$$

$$\lim_{n\to\infty}\cup_n=\lim_{n\to\infty}\frac{1}{n+cosx}=0\ \lim_{n\to\infty}u_n=\lim_{n\to\infty}\frac{1}{n+\cos x}=0.$$

Therefore, the given series also coincides. Therefore, the sum of the series is measurable according to Lebesgue.

# 13 LEBESGUE INTEGRAL OF BOUNDED FUNCTION

***Basic theoretical statement***

*Calculation of the Lebesgue integral.*

Riemann integration criterion of function $f(x)$. In order for a Riemann function $f(x)$ to be integrated, it is necessary and sufficient that it be continuous almost everywhere on the segment $[a;b]$.

Relation of Riemann and Lebesgue integrals. If a Riemann function $f(x)$ is integrated on a segment $[a;b]$, then it is also integrated on Lebesgue on this segment, and both integrals are equal:

$$(R)\int_a^b f(x)dx = (L)\int_a^b f(x)dx.$$

***Ex. 13.1.*** Calculate the Lebesgue integral:

$$\int_{[3;3]} signcos\Pi x\, dm \int_{-[3;3]} sign\cos \pi x dx$$

$$sign\, t = \begin{cases} 1 \text{ при } t > 0 \\ -1 \text{ при } t > 0 \\ 0 \quad t = 0 \end{cases} \begin{cases} 1 \text{ при } t > 0 \\ -1 \text{ при } t > 0 \\ 0 \quad t = 0 \end{cases}$$

Solution: Consider the case $\cos \pi x > 0$:

$$\pi x \in \left(-\frac{\pi}{2};\frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2};\frac{5\pi}{2}\right) \cup \left(-\frac{5\pi}{2};-\frac{3\pi}{2}\right).$$

Consider the case $\cos \pi x < 0$:

$$\pi x \in \left(-\frac{3\pi}{2};-\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2};\frac{3\pi}{2}\right) \cup \left(\frac{5\pi}{2};-\frac{7\pi}{2}\right) \cup \left(-\frac{7\pi}{2};-\frac{5\pi}{2}\right).$$

Consider the case $\cos \pi x = 0$:

$$\pi x = -\frac{5\pi}{2};-\frac{3\pi}{2};-\frac{\pi}{2};\frac{\pi}{2};\frac{3\pi}{2};\frac{5\pi}{2},$$

$$x = 0 \quad A_1 = \left\{-\frac{5}{2};-\frac{3}{2};-\frac{1}{2};\frac{1}{2};\frac{3}{2};\frac{5}{2}\right\},$$

$$x > 0 \quad A_2 = \left(-\frac{5}{2};-\frac{3}{2}\right) \cup \left(\frac{5}{2};\frac{3}{2}\right) \cup \left(\frac{1}{2};-\frac{1}{2}\right),$$

$$x<0 \quad A_3=\left(-3;-\frac{5}{2};\right).$$

So,

$$(L)\int_{-3}^{3} sign\cos\pi x dx = 0\cdot m(A_1)+1\cdot m(A_2)-1\cdot m(A_3)=0.$$

***Ex. 13.2.*** Calculate the Lebesgue integral of the function $f(x)=\sin x$ on a segment $\left[0;\frac{\pi}{2}\right].$

Solution:

$$(L)\int_0^{\frac{\pi}{2}} \sin x dx = -\cos\frac{\pi}{2}+\cos 0=1.$$

***Ex. 13.3.*** Calculate the Lebesgue integral of the function

$$f(x)=\begin{cases}\sin x, & x-\textit{раціональне},\\ \cos x, & x-\textit{ірраціональне}\end{cases}$$

on a segment $\left[0;\frac{\pi}{2}\right].$

Solution: We use the theorem on equivalent functions. We have: $f(x)$ $\sim \cos x$, $x-$ ирац.$\sim \cos x, x-$ ирац, if $x$ is irrational. So,

$$(L)\int_0^{\frac{\pi}{2}} \cos x dx = \sin\frac{\pi}{2}-\sin 0=1.$$

***Exercises for independent work***

***Ex. 13.4.*** Calculate the Lebesgue integral of the function

$$f(x)=\begin{cases}\sin x, & x-\textit{раціональне},\\ \sin^2 x, & x-\textit{ірраціональне}\end{cases}$$

on a segment $\left[0;\frac{\pi}{2}\right].$

## 14 EXAMPLES OF TASKS OF INCREASED COMPLEXITY

***Ex. 14.1.*** Prove that inequalities hold for any four points $x, y, z, t$ of the metric space $(X, \rho)$

1. $|\rho(x,z) - \rho(y,z)| \le \rho(x,y)$.
2. $|\rho(x,z) - \rho(y,t)| \le \rho(x,y) + \rho(z,t)$.

***Ex. 14.2.*** Prove that the axioms of metric space are equivalent to the following two axioms:

1. $\rho(x,y) = 0 \Leftrightarrow x = y$.
2. $\rho(x,z) + \rho(y,z) \ge \rho(x,y)$.

***Ex. 14.3.*** Let $\rho(x,y)$ be the metric on the set $X$. Prove that the functions:

1. $\rho_1(x,y) = \dfrac{\rho(x,y)}{(1+\rho(x,y))}$
2. $\rho_2(x,y) = \min\{\rho(x,y), 1\}$

are metrics on $X$.

***Ex. 14.4.*** Prove that the specified sets with given metrics are complete metric spaces:

1. The set $l^\infty$ of all bounded numerical sequences $x = (x_1, x_2, ...)$ that satisfy condition $\sum_{i=1}^{\infty} x_i < \infty$ with metrics $\rho(x,y) = \sup_k |x_k - y_k|$.

2. The set $l^\infty$ of all numerical sequences $x = (x_1, x_2, ...)$ that tend to zero, with a metric $\rho(x,y) = \max_k |x_k - y_k|$.

3. The set $C_{[a;b]}$ of all continuous functions on a segment $[a;b]$ with a metric $\rho(x,y) = \max\limits_{t \in [a;b]} |x(t) - y(t)|$.

***Ex. 14.5.*** Are such functionalities linear in space $C_{[0;1]}$? In space $L_{2[0;1]}$?

1. $F(x) = \int_0^1 x(t) \cdot \sin t dt$, $F(x) = \dfrac{x}{2}$.
2. $F(x) = \int_0^1 x(t) \cdot sign\left(t - \dfrac{1}{2}\right) dt$, $F(x) = \int_0^1 x(t^2) \cdot t^{-2} dt$.

***Ex. 14.6.*** Which of the following operators are continuous?

1. $A: R^n \to R^n$ defined by the formula $y_i = \sum_{k=1}^{n} a_{ik}\tau_k, i = 1,\ldots,n;$
2. $A: C[0,1] \to C[0,1]$ defined by the formula $Ax(t) = \int_0^1 x(\tau)d\tau;$
3. $\frac{d}{dt}: C^1[0,1] \to C[0,1]$.

# LITERATURE

1. John B. Conway. A Course in Functional Analysis / John B. Conway. – Springer-Verlag New York, 2007. – 400. DOI: 10.1007/978-1-4757-4383-8.

2. Haim Brezis. Functional Analysis, Sobolev Spaces and Partial Differential Equations / Haim Brezis. Springer Science+Business Media, LLC, 2011. – 600. DOI: 10.1007/978-0-387-70914-7.

3. Elias M. Stein. Functional Analysis: Introduction to Further Topics in Analysis (Princeton Lectures in Analysis) / Elias M. Stein, Rami Shakarchi. – Princeton University Press, 2011. – 448.

*Навчальне видання*

# ФУНКЦІОНАЛЬНИЙ АНАЛІЗ

**Методичні вказівки**

**до занять та самостійної роботи**

*(англійською мовою)*

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